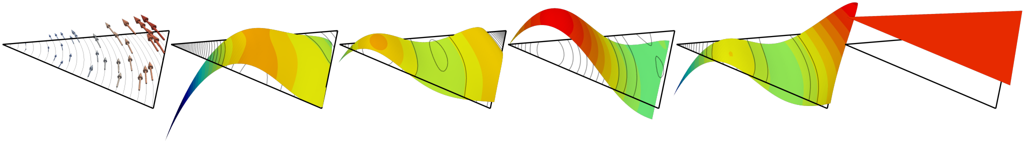
Embedded Trefftz discontinuous Galerkin methods

Christoph Lehrenfeld, Paul Stocker

Georg-August-Universität, Göttingen, Germany



Trefftz-DG methods

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, \\ + \text{ bndc. on } \partial\Omega, \end{cases}$$

Find $u_h \in \mathbb{T}^p(\mathcal{T}_h)$, s.t. $a_h^{\mathsf{DG}}(u_h, v_h) = \ell_h^{\mathsf{DG}}(v_h) \qquad \forall v_h \in \mathbb{T}^p(\mathcal{T}_h)$ with

$$\mathbb{T}^p(\mathcal{T}_h) := \Pi_{K\in\mathcal{T}_h}\mathbb{T}^p(K), \quad \mathbb{T}^p(K) := \{v_h \in \mathbb{P}^p(K) \text{ s.t. } \mathcal{L}u = 0 \text{ on } K\}$$

$$\dim \mathbb{T}^p(\mathcal{T}_h) = \mathcal{O}(p^{n-1}) \ll \dim \mathbb{P}^p(\mathcal{T}_h) = \mathcal{O}(p^n)$$

Assumption: Trefftz space is 'rich enough' ($\mathcal{L} = \sum_{l=1}^d \alpha_l \partial_{x_l}^{\beta_l}$ for $\alpha_l \in \mathbb{R}$ and $\beta_l \in \mathbb{N}$)

E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, Proc. 2nd Int. Cong. Appl. Mech., Zurich, 1926

Example: Laplace equation

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

$$\mathbb{T}^p(K) = \{1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \dots\}$$

F. Li, C.-W. Shu, *A local-structure-preserving local discontinuous Galerkin method for the Laplace equation*, Methods Appl. Anal., 2006

R. Hiptmair, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [...] of Trefftz hp-dGFEM*, ESAIM Math. Model. Num. Anal., 2014

Example: Laplace equation

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

$$\mathbb{T}^p(K) = \{1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \dots\}$$

$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\{\partial_{\mathbf{n}} u\}\!\}[\![v]\!]}^{\text{consistency}} \overbrace{-\{\!\{\partial_{\mathbf{n}} v\}\!\}[\![u]\!]}^{\text{symmetry}} \overbrace{+ \alpha p^2 h^{-1} [\![u]\!][\![v]\!]}^{\text{stability}} \, ds$$

$$+ \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F -\partial_{\mathbf{n}} u \, v - \partial_{\mathbf{n}} v \, u + \alpha p^2 h^{-1} uv \, ds$$

$$\ell(v) = \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F (-\partial_{\mathbf{n}} v + \alpha p^2 h^{-1} v) g \, ds.$$

$\{\!\{\cdot\}\!\}$: average across facets, $[\![\cdot]\!]$: jump across facets. $\rightsquigarrow$ communication between neighbors.

F. Li, C.-W. Shu, *A local-structure-preserving local discontinuous Galerkin method for the Laplace equation*, Methods Appl. Anal., 2006

R. Hiptmair, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [...] of Trefftz hp-dGFEM*, ESAIM Math. Model. Num. Anal., 2014

Embedded Trefftz-DG method

- Goal: Represent Trefftz basis $\{\psi_j\}_M$ in the standard DG basis $\{\phi_i\}_N$:

$$\psi_j = \sum_{j=1}^{N} \mathbf{T}_{ij}\phi_i, \; j = 1, .., M, \text{ for } \mathbf{T} \in \mathbb{R}^{N\times M}.$$

Then instead of solving $\mathbf{A}\mathbf{u}_{\mathbb{P}} = \mathbf{b}$ we solve $\mathbf{T}^T\mathbf{A}\mathbf{T}\;\mathbf{u}_{\mathbb{T}} = \mathbf{T}^T\mathbf{b}$.

Embedded Trefftz-DG method

- Goal: Represent Trefftz basis $\{\psi_j\}_M$ in the standard DG basis $\{\phi_i\}_N$:

$$\psi_j = \sum_{j=1}^{N} \mathbf{T}_{ij}\phi_i, \; j = 1, .., M, \text{ for } \mathbf{T} \in \mathbb{R}^{N\times M}.$$

  Then instead of solving $\mathbf{A}\mathbf{u}_{\mathbb{P}} = \mathbf{b}$ we solve $\mathbf{T}^T\mathbf{A}\mathbf{T}\;\mathbf{u}_{\mathbb{T}} = \mathbf{T}^T\mathbf{b}$.
- Recipe:

$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j\rangle_K$$

Embedded Trefftz-DG method

- Goal: Represent Trefftz basis $\{\psi_j\}_M$ in the standard DG basis $\{\phi_i\}_N$:

$$\psi_j = \sum_{j=1}^{N} \mathbf{T}_{ij}\phi_i, \; j = 1, .., M, \text{ for } \mathbf{T} \in \mathbb{R}^{N\times M}.$$

  Then instead of solving $\mathbf{A}\mathbf{u}_{\mathbb{P}} = \mathbf{b}$ we solve $\mathbf{T}^T\mathbf{A}\mathbf{T}\;\mathbf{u}_{\mathbb{T}} = \mathbf{T}^T\mathbf{b}$.
- Recipe:

$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j\rangle_K$$

- Benefit: $\mathbf{T}^T\mathbf{A}\mathbf{T} \in \mathbb{R}^{M\times M}$ with

$$\dim \mathbb{T}^p(\mathcal{T}_h) = M \ll N = \dim \mathbb{P}^p(\mathcal{T}_h)$$

Embedded Trefftz-DG method

$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K$$

On each mesh element use SVD (or QR)

$$\mathbf{W}\big|_K = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_L & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} — & \mathbf{v}_1^T & — \\ & \vdots & \\ — & \mathbf{v}_L^T & — \\ — & \mathbf{v}_{L+1}^T & — \\ & \vdots & \\ — & \mathbf{v}_N^T & — \end{pmatrix}$$

Embedded Trefftz-DG method

$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K$$

On each mesh element use SVD (or QR)

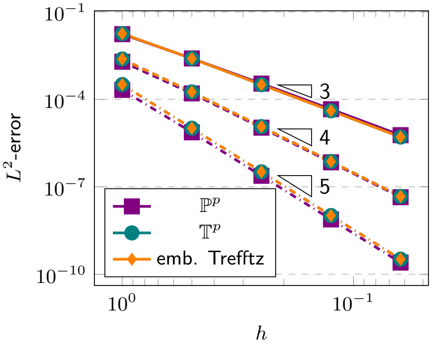
$$\mathbf{W}\big|_K = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

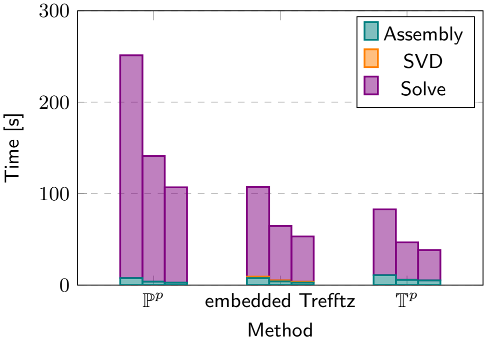
Lemma (Conditioning of the embedded Trefftz method)

$$\kappa_2(\mathbf{T}^T\mathbf{A}\mathbf{T}) \leq \kappa_2(\mathbf{A}).$$

Example: Laplace equation II

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$





Benefits of Embedded Trefftz DG

So far:

Embedded Trefftz DG ...

1. ... facilitates implementation of existing polynomial Trefftz methods
2. ... is computationally (a bit) more expensive than "direct" Trefftz spaces
3. ... inherites conditioning properties from DG scheme

Next up:

Can we deal with ...

1. ... PDEs where no (suitable) polynomial Trefftz spaces exists?
2. ... inhomogeneous PDEs?

Embedded Trefftz - no polynomial Trefftz space

$$\begin{cases} \mathcal{L}u = -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$

Trefftz space given by plane waves

$$\{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ with } j = -p, \ldots, p\} \not\subset \mathbb{P}^p(\mathcal{T}_h).$$

O. Cessenat, B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz problem*, SIAM J. Numer. Anal. 1998,

R. Hiptmair, A. Moiola, I. Perugia, *A Survey of Trefftz Methods for the Helmholtz Equation*, Lect. Notes Comput. Sci. Eng., 2016,

Embedded Trefftz - no polynomial Trefftz space

$$\begin{cases} \mathcal{L}u = -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$

Introduce a *weak Trefftz space* for the embedding

$$\mathbb{T}^p(\mathcal{T}_h) = \{v \in \mathbb{P}^p(\mathcal{T}_h),\ \Pi\mathcal{L}v = 0 \text{ on each } K \in \mathcal{T}_h\}.$$

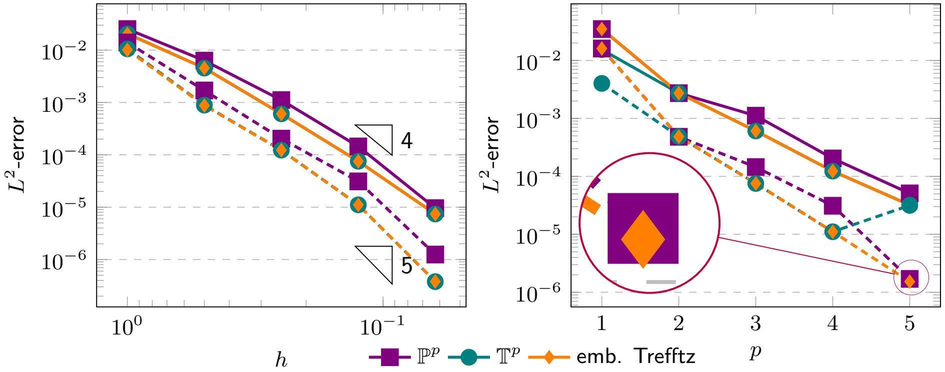
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \psi_j \rangle_K, \quad \forall \psi_j \in \mathbb{P}^{p-2}$$

O. Cessenat, B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz problem*, SIAM J. Numer. Anal. 1998,

R. Hiptmair, A. Moiola, I. Perugia, *A Survey of Trefftz Methods for the Helmholtz Equation*, Lect. Notes Comput. Sci. Eng., 2016,

Example: Helmholtz

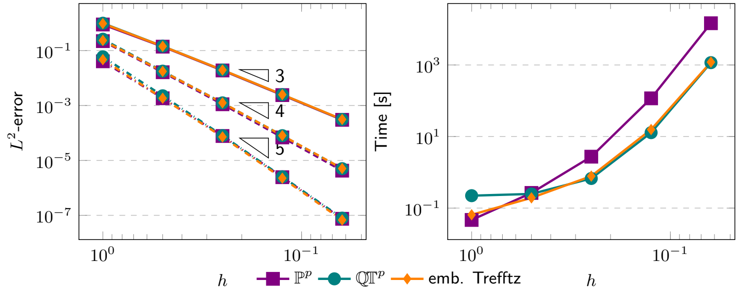
$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$



Example: Acoustic wave equation

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + c(\mathbf{x})^{-2} \frac{\partial v}{\partial t} = 0 & \text{in } \Omega \times [0,T], \\ \nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } \Omega \times [0,T], \\ v(\cdot,0) = v_0, \ \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \times \{0\}, \\ v = g_D & \text{on } \partial\Omega \times [0,T], \end{cases}$$

With $c(x,y) = 1 + x + y$



L.-M. Imbert-Gérard, Andrea Moiola, PS, *A space-time quasi-Trefftz DG method for the wave equation [...]* , arXiv preprint, arXiv:2011.04617, 2021

Embedded Trefftz - inhomogeneous problem

$$\begin{cases} \mathcal{L}u = f \text{ in } \Omega, \\ + \text{ bndc. on } \partial\Omega, \end{cases}$$

On each element we can construct a particular solution using the pseudo-inverse

$$\mathbf{W}^\dagger = \begin{pmatrix} | & & | & | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_L & \mathbf{v}_{L+1} & \dots & \mathbf{v}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{\sigma_L} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{u}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_L^T & \text{---} \\ \text{---} & \mathbf{u}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_N^T & \text{---} \end{pmatrix}$$

For $u_{h,f}$ a particular solution, we are looking for a solution $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$ that (uniquely) solves

$$a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) - a_h(u_{h,f}, v_{\mathbb{T}}) \quad \forall\, v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h).$$

Q. Hu, L. Yuan, *A plane wave method combined with local spectral elements for nonhomogeneous Helmholtz equation [...]* , Adv. Comput. Math., 2018,

A. Uściłowska-Gajda, et al., *Comparison of two types of Trefftz method for the solution of inhomogeneous elliptic problems*, Comput. Assist. Mech. Eng. Sci., 2003,

Example: Possion

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g_D \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

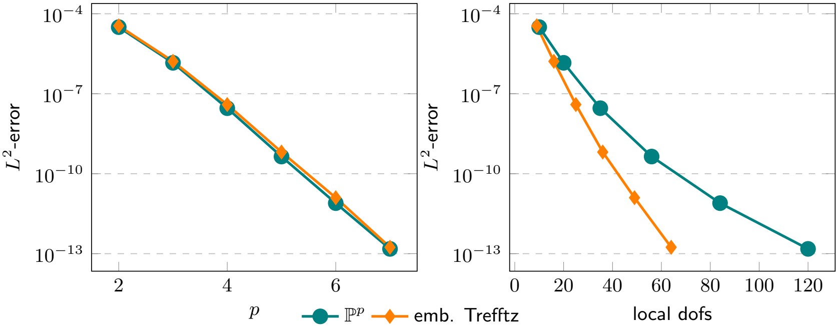


Figure: Numerical results for Poisson equation, 3D, mesh size $h = 0.25$.

Example code

Require: Basis functions $\{\phi_i\}_i$, DG formulation (a_h, l), operators $\mathcal{L}$, $\tilde{\mathcal{L}}$, truncation parameter ε, r.h.s. f

function DG MATRIX
  $(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i)$
  $(\mathbf{l})_i = \ell(\phi_i)$
for $K \in \mathcal{T}_h$ **do**
  $(\mathbf{W}_K)_{ij} = \langle \mathcal{L}\phi_j, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
  $\mathbf{T}_K = \ker_h(\varepsilon; \mathbf{W}_K)$
  if $f \neq 0$ **then**
    $(\mathbf{w}_K)_i = \langle f, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
    $(\mathbf{u}_f)_K = \mathbf{W}_K^{\dagger} \mathbf{w}_K$
Solve $\mathbf{T}^T \mathbf{A} \mathbf{T}\ \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T (\mathbf{l} - \mathbf{A}\mathbf{u}_f)$
$\mathbf{u}_h = \mathbf{T}\mathbf{u}_{\mathbb{T}} + \mathbf{u}_f$
output $\mathbf{u}_h$

```
def Solve(mesh, order, dgscheme,
                    L, Ltilde, eps,
                    rhs):
    fes = L2(mesh,order=order,dgjumps=True)
    uh = GridFunction(fes)
    a,f = dgscheme(fes)
    u,v = fes.TnT()
    W = L(u)*Ltilde(v)*dx
    w = rhs*Ltilde(v)*dx
    T, uf = TrefftzEmbedding(W,fes,eps,w)
    Tt = T.CreateTranspose()
    TA = Tt@a.mat@T
    ut = TA.Inverse()*(Tt*(f.vec-a.mat*uf))
    uh.vec.data = T*ut + uf
    return uh
```

Conclusion

Summary

- reduce test/trial-spaces using a projection that infers structural properties
- construct an embedding of Trefftz (like) subspaces in a very generic way
- works for inhomogeneous PDEs and non-constant coefficients

References

R. Hiptmair, A. Moiola, and I. Perugia.
A survey of Trefftz methods for the Helmholtz equation.
Lect. Notes Comput. Sci. Eng. Springer, 2016.

E. Kita and N. Kamiya.
Trefftz method: an overview.
Adv. Eng. Softw., 24(1–3):3–12, 1995.

C. Lehrenfeld and P. Stocker.
Embedded Trefftz discontinuous Galerkin methods.
arXiv preprint, arXiv:2201.07041, 2022.

Q.-H. Qin.
Trefftz finite element method and its applications.
Appl. Mech. Rev., 58(5):316–337, 2005.

E. Trefftz.
Ein Gegenstück zum Ritzschen Verfahren.
Proc. 2nd Int. Cong. Appl. Mech., Zurich, 1926, pages 131–137, 1926.

paulst.github.io/NGSTrefftz