Numerical treatment of the vectorial equations of solar oscillations

Martin Halla, Christoph Lehrenfeld, Paul Stocker

Georg-August-Universität, Göttingen, Germany

Galbrun's equation

Galbrun's equation for time-harmonic acoustic waves for the unknown $\mathbf{u}$ is given by the partial differential equation

$$\rho(-i\omega + \partial_{\mathbf{b}})^2\mathbf{u} - \operatorname{grad}(\rho c_s^2 \operatorname{div} \mathbf{u}) - i\omega\gamma\rho\mathbf{u} \\ + \operatorname{Hess}(p)\mathbf{u} + (\operatorname{div} \mathbf{u}) \operatorname{grad} p - \operatorname{grad}(\operatorname{grad} p \cdot \mathbf{u}) = \mathbf{f}$$

in the presence of density ρ, pressure p, sound speed c_s, background velocity $\mathbf{b}$, damping coefficient γ, and source $\mathbf{f}$.

$$\partial_{\mathbf{b}} := \mathbf{b} \cdot \nabla$$

$\mathbf{b}, c_s, \rho, : \mathcal{O} \to \mathbb{R}$ be continuous with

$$\underline{c_s} \leq c_s \leq \overline{c_s}, \qquad \underline{\rho} \leq \rho \leq \overline{\rho},$$

$$\left\| c_s^{-1}\mathbf{b} \right\|_\infty^2 \leq c < 1, \qquad \operatorname{div}(\rho\mathbf{b}) = 0 \text{ in } \mathcal{O}.$$

and boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}$$

Short excerpt of the Literature

1931, H. Galbrun, *Propagation d'une onde sonore dans l'atmosphre et théorie des zones de silence*

⋮

2020, M. Maeder , G. Gabard and S. Marburg *90 Years of Galbrun's Equation: An Unusual Formulation for Aeroacoustics and Hydroacoustics in Terms of the Lagrangian Displacement*

⋮

2003, A.-S. Bonnet-Ben Dhia et at., *Regularization of the time-harmonic Galbrun's equations*

2018, J. Chabassier, M. Duruflé, *Solving time-harmonic Galbrun's equation with an arbitrary flow. Application to Helioseismology*

2019, L. Hägg and M. Berggren *On the well-posedness of Galbrun's equation*

2021, H. Barucq et al., Outgoing modal solutions for Galbrun's equation in helioseismology

2021, M. Halla and T. Hohage, *On the well-posedness of the damped time-harmonic Galbrun equation and the equations of stellar oscillations,*

⋮

Recap: Well-posedness for Galbrun's equation, without pressure

M. Halla and T. Hohage, *On the well-posedness of the damped time-harmonic Galbrun equation and the equations of stellar oscillations*, SIAM J. Math. Anal., 53 (2021), pp. 4068–4095

A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Math., 92 (2002), pp. 679–710

A. Buffa, *Remarks on the discretization of some noncoercive operator with applications to heterogeneous maxwell equations*, SIAM Journal on Numerical Analysis, 43 (2005), pp. 1–18

Constant pressure

Find $\mathbf{u} \in \mathbb{X}$ such that

$$a(\mathbf{u}, \mathbf{u}') = \langle \mathbf{f}, \mathbf{u}' \rangle \quad \forall \mathbf{u}' \in \mathbb{X}$$

with the sesquilinear form $a(\mathbf{u}, \mathbf{u}')$

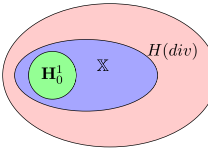
$$\langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}})\mathbf{u}, (\omega + i\partial_{\mathbf{b}})\mathbf{u}' \rangle - i\omega \langle \gamma \rho \mathbf{u}, \mathbf{u}' \rangle.$$

where

$$\mathbb{X} = \{\mathbf{u} \in L^2(\mathcal{O}, \mathbb{C}^3) : \ \operatorname{div} \mathbf{u} \in L^2(\mathcal{O}), \ \partial_{\mathbf{b}} \mathbf{u} \in L^2(\mathcal{O}, \mathbb{C}^3), \ \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \partial\mathcal{O}\}$$

and inner product

$$\langle \mathbf{u}, \mathbf{u}' \rangle_{\mathbb{X}} := \langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}' \rangle + \langle \partial_{\mathbf{b}} \mathbf{u}, \partial_{\mathbf{b}} \mathbf{u}' \rangle + \langle \mathbf{u}, \mathbf{u}' \rangle$$



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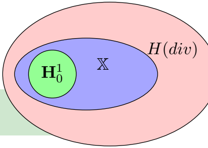

Pitfall! $\mathbb{X} \not\hookrightarrow L^2$, problem is not weakly coercive!

Weak T-coercive

Definition

The problem

$$Au = f$$

is weak T-coercive if there exists $T \in L(X)$ bounded bijective, $K \in L(X)$ compact, $B \in L(X)$ coercive

$$AT = B + K.$$

- If A is weakly T-coercive, then A is Fredholm operator with index zero.
- If A is a Fredholm operator, then A is injective $\Leftrightarrow$ A surjective

Space decomposition

Sesquilinear form $a(\mathbf{u}, \mathbf{u}')$ given by

$$\langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}})\mathbf{u}, (\omega + i\partial_{\mathbf{b}})\mathbf{u}' \rangle - i\omega \langle \gamma \rho \mathbf{u}, \mathbf{u}' \rangle.$$

with a splitting

$$\mathbb{X} = V \oplus W$$

and for $\mathbf{u} = \mathbf{v} + \mathbf{w}$

$$T\mathbf{u} = \mathbf{v} - \mathbf{w}$$

Idea: Let $A \in L(\mathbb{X})$ be the operator associated to $a(\cdot, \cdot)$

$$A \leftrightarrow \begin{pmatrix} A_{\mathbf{vv}} & A_{\mathbf{vw}} \\ A_{\mathbf{wv}} & A_{\mathbf{ww}} \end{pmatrix}$$

$A_{\mathbf{vv}}, A_{\mathbf{vw}}, A_{\mathbf{wv}}$ with compact part
$A_{\mathbf{ww}}$ with good sign

Wishlist space V

- $V \hookrightarrow L^2$
- $\|\operatorname{div} \mathbf{v}\|_{L^2} \geq \|\mathbf{v}\|_{\mathbb{X}}$

Space decomposition

$$\mathbb{X} = V \oplus W$$

with

$$V := \{\mathbf{u} \in \mathbf{H}_0^1 \colon \langle \nabla \mathbf{u}, \nabla \mathbf{u}' \rangle = 0 \text{ for all } \mathbf{u}' \in \mathbf{H}_0^1 \text{ with } \operatorname{div} \mathbf{u}' = 0\},$$
$$W := \{\mathbf{u} \in \mathbb{X} \colon \operatorname{div} \mathbf{u} = 0\}$$

Assumption: There exists a constant $\beta > 0$ such that

$$\inf_{f \in L_0^2 \setminus \{0\}} \sup_{\mathbf{u} \in \mathbb{X} \setminus \{0\}} \frac{|\langle \operatorname{div} \mathbf{u}, f \rangle|}{\|\nabla \mathbf{u}\|_{L^2} \|f\|_{L^2}} > \beta$$

- $V \hookrightarrow L^2$
- $\|\operatorname{div} \mathbf{v}\|_{L^2} \geq \|\mathbf{v}\|_{\mathbb{X}}$

Discretization and T-compatability

T. Hohage and L. Nannen, *Convergence of infinite element methods for scalar waveguide problems*, BIT, 55 (2015), pp. 215–254

A.-S. Bonnet-BenDhia, C. Carvalho, and P. Ciarlet, *Mesh requirements for the finite element approximation of problems with sign-changing coefficients*, Numerische Mathematik, 138 (2018), pp. 801–838

M. Halla, *Galerkin approximation of holomorphic eigenvalue problems: weak T-coercivity and T-compatibility*, Numer. Math., 148 (2021), pp. 387–407

Conforming discretization

We consider finite element spaces

$$\mathbb{X}_n := \{\mathbf{u} \in \mathbf{H}^1 \colon \boldsymbol{\nu} \cdot \mathbf{u} = 0, \mathbf{u}|_\tau \in (P_k(\tau))^3 \quad \forall \tau \in \mathcal{T}_n\},$$

The approximated problem then reads:

$$\text{find } \mathbf{u}_n \in \mathbb{X}_n \text{ s.t.} \quad a(\mathbf{u}_n, \mathbf{u}_n') = \langle \mathbf{f}, \mathbf{u}_n' \rangle \quad \forall \mathbf{u}_n' \in \mathbb{X}_n.$$

with the sesquilinear form $a(\mathbf{u}_n, \mathbf{u}_n')$ given by

$$\langle c_s^2 \rho \operatorname{div} \mathbf{u}_n, \operatorname{div} \mathbf{u}_n' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}})\mathbf{u}_n, (\omega + i\partial_{\mathbf{b}})\mathbf{u}_n' \rangle - i\omega \langle \gamma \rho \mathbf{u}_n, \mathbf{u}_n' \rangle$$

Let $A \in L(\mathbb{X})$ be the operator associated to $a(\cdot, \cdot)$ and $A_n := P_{\mathbb{X}_n} A|_{\mathbb{X}_n}$.

New T-compatibility condition

Theorem:

- There exists $p_n \in L(X, X_n)$ s.t. $\|p_n u\|_{X_n} \to \|u\|_X \, \forall u \in X$
- The discrete system can be written as

$$A_n T_n = B_n + K_n.$$

- $B_n, T_n \in L(X_n)$ bijective, $K_n \in L(X_n)$ with $(K_n)_{n\in\mathbb{N}}$ is compact
- A_n, B_n, T_n *asymptotic consistent*, i.e.

$$\lim_{n\to\infty} \|T_n p_n u - p_n T u\|_{X_n} = 0 \text{ for each } u \in X,$$

Then

$$A_n \text{ is invertible and } u_n \to u$$

Discrete decomposition

$$\mathbb{X}_n = V_n \oplus W_n, \qquad T_n \mathbf{u}_n = \mathbf{v}_n - \mathbf{w}_n$$

with

$$W_n := \{\mathbf{u}_n \in \mathbb{X}_n \colon \operatorname{div} \mathbf{u}_n = 0\}$$
$$V_n := \{\mathbf{u}_n \in \mathbb{X}_n \cap \mathbf{H}_0^1 \colon \langle \nabla \mathbf{u}_n, \nabla \mathbf{u}_n' \rangle = 0 \quad \forall \mathbf{u}_n' \in \mathbf{H}_0^1 \cap W_n\},$$

Lemma

Assume discrete inf-sup stability $(k > d)$*. For each* $\mathbf{u} \in \mathbb{X}$ *it holds* $\lim_{n\to\infty} \|T_n P_{\mathbb{X}_n} \mathbf{u} - P_{\mathbb{X}_n} T \mathbf{u}\|_{\mathbb{X}} = 0$

The following problem is well-posed:

$$\text{Find } \mathbf{v}_n \in V_n \text{ such that } \operatorname{div} \mathbf{v}_n = \operatorname{div} \mathbf{u}_n,$$

Let $D_n^{-1} \in L(Q_n, V_n)$ be the respective solution operator. Setting $\mathbf{w}_n := \mathbf{u}_n - \mathbf{v}_n$ it is sufficient to show

$$\lim_{n\to\infty} \left\| \mathbf{v} - D_n^{-1} P_{Q_n} \operatorname{div} \mathbf{v} \right\|_{\mathbf{H}^1} = 0$$

Towards heterogeneous pressure

Towards heterogeneous pressure

Consider $a(\mathbf{u}, \mathbf{u}')$ with $\mathbf{q} = (c_s^2 \rho)^{-1} \nabla p$

$$\langle c_s^2 \rho (\operatorname{div} + \mathbf{q} \cdot) \mathbf{u}, (\operatorname{div} + \mathbf{q} \cdot) \mathbf{u}' \rangle - \langle \rho (\omega + i \partial_{\mathbf{b}}) \mathbf{u}, (\omega + i \partial_{\mathbf{b}}) \mathbf{u}' \rangle$$
$$- i\omega \langle \rho \gamma \mathbf{u}, \mathbf{u}' \rangle + \langle (\operatorname{Hess}(p) - c_s^2 \rho \, \mathbf{q} \otimes \mathbf{q}) \mathbf{u}, \mathbf{u}' \rangle.$$

Consider now the divergence operator $D \in L(V, L^2_0), D\mathbf{v} := \operatorname{div} \mathbf{v}$. We know that $D^{-1} \in L(L^2_0, V)$. Now consider

$$\tilde{D} \in L(V, L^2_0), \; \tilde{D}\mathbf{v} := D\mathbf{v} + \mathbf{q} \cdot \mathbf{v} + M\mathbf{v} + F\mathbf{v}$$

with $M\mathbf{v} := -\operatorname{mean}(\mathbf{q} \cdot \mathbf{v})$ and a finite dimensional operator $F\mathbf{v} := \sum_{n=1}^{N} \phi_n \langle \operatorname{div} \mathbf{v}, \operatorname{div} \psi_n \rangle$, $\phi_n \in L^2_0, \psi_n \in V$ such that $\tilde{D}$ is bijective.

Towards heterogeneous pressure

For $\mathbf{u} \in \mathbb{X}$ we construct a topological decomposition as follows.

$$\mathbf{v} := \tilde{D}^{-1}\tilde{D}\mathbf{u} \qquad \text{and} \qquad \mathbf{w} := \mathbf{u} - \mathbf{v}.$$

Further it follows that

$$\begin{aligned}(\mathrm{div} + \mathbf{q}\cdot)\mathbf{w} &= (\mathrm{div} + \mathbf{q}\cdot)\mathbf{u} - (\mathrm{div} + \mathbf{q}\cdot)\mathbf{v} \\ &= (\mathrm{div} + \mathbf{q}\cdot)\mathbf{u} - (\mathrm{div} + \mathbf{q}\cdot + M + F)\mathbf{v} + (M + F)\mathbf{v} \\ &= -(M + F)\mathbf{w}\end{aligned}$$

is compact.

Numerical examples

Numerical example

$$\rho = 1.5 + 0.2\cos(\pi x/4)\sin(\pi y/2), \quad c^2 = 1.44 + 0.16\rho, \quad \omega = 0.78 \times 2\pi,$$

$$\mathbf{b} = \frac{\text{coeff}}{\rho}\begin{pmatrix} 0.3 + 0.1\cos(\pi y/4) \\ 0.2 + 0.08\sin(\pi x/4) \end{pmatrix}, \quad \gamma = 0.1, \quad p = 1.44\rho + 0.08\rho^2$$

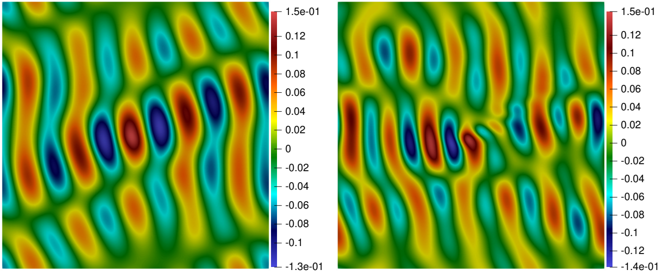

Figure: The real part of the first entry of the reference solution computed with $p = 5$ and $h = 2^{-4}$ for two different values of the coefficient of the flow field $\mathbf{b}$, coeff $= 0.2$ on the left and coeff $= 1.5$ on the right.

J. Chabassier and M. Duruflé, *Solving time-harmonic Galbrun's equation with an arbitrary flow. Application to Helioseismology*, Research Report RR-9192, INRIA Bordeaux, July 2018

Assumption: There exists a constant $\beta > 0$ such that

$$\inf_{f_n \in Q_n \setminus \{0\}} \sup_{\mathbf{u}_n \in \mathbb{X}_n \setminus \{0\}} \frac{|\langle \operatorname{div} \mathbf{u}_n, f \rangle|}{\|\nabla \mathbf{u}_n\|_{L^2} \|f_n\|_{L^2}} > \beta_n$$

- $k \geq 4$ and no singular points in the mesh
- $k \geq 2$ and barycentric mesh refinement

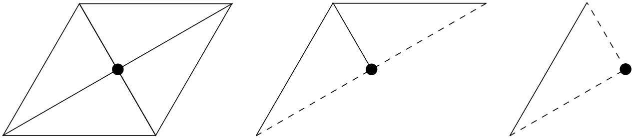

Figure: Examples of singular mesh points, dashed lines denote boundary of the domain.

Assumption 2: Sub-sonic flow: $\|c_0^{-1}\mathbf{b}\|_{L^\infty} < \beta_h \frac{\underline{\underline{c_0}}^2 \underline{\underline{\rho}}}{\overline{\overline{c_0}}^2 \overline{\overline{\rho}}}$.

L. R. Scott and M. Vogelius, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials*, RAIRO, Modélisation Math. Anal. Numér., 19 (1985), pp. 111–143

J. Guzmán and M. Neilan, *Inf-sup stable finite elements on barycentric refinements producing divergence-free approximations in arbitrary dimensions*, SIAM J. Numer. Anal., 56 (2018), pp. 2826–2844

Meshes

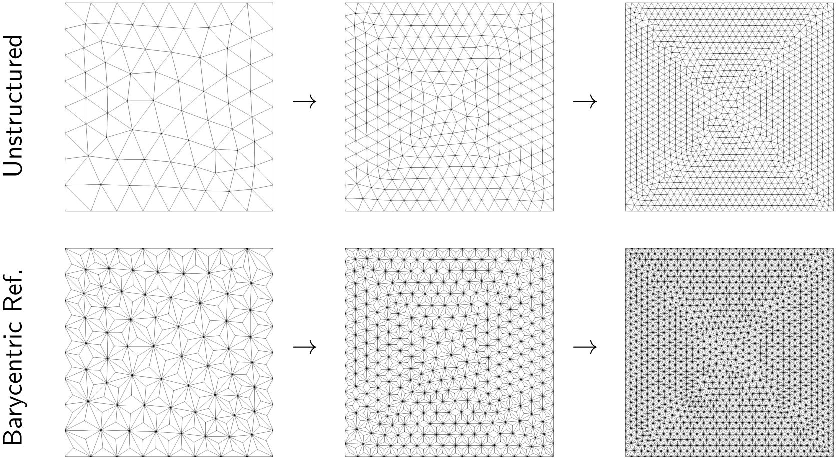

Figure: Meshes

$k = 2$

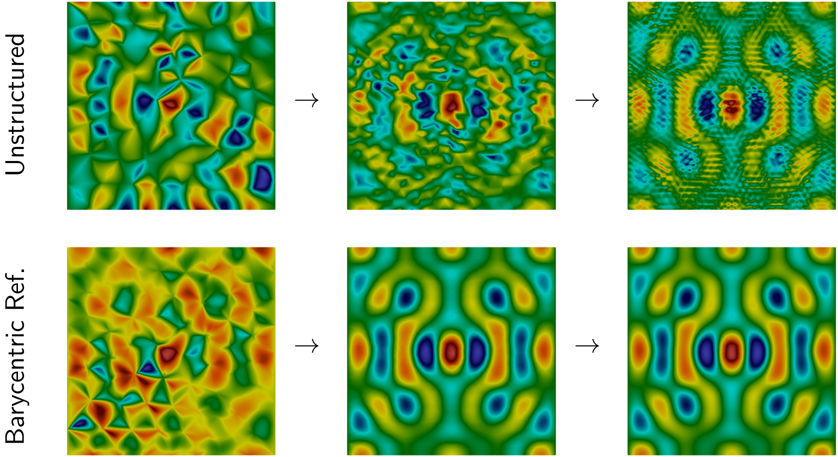

Figure: Solutions for $k = 2$.

$k = 4$

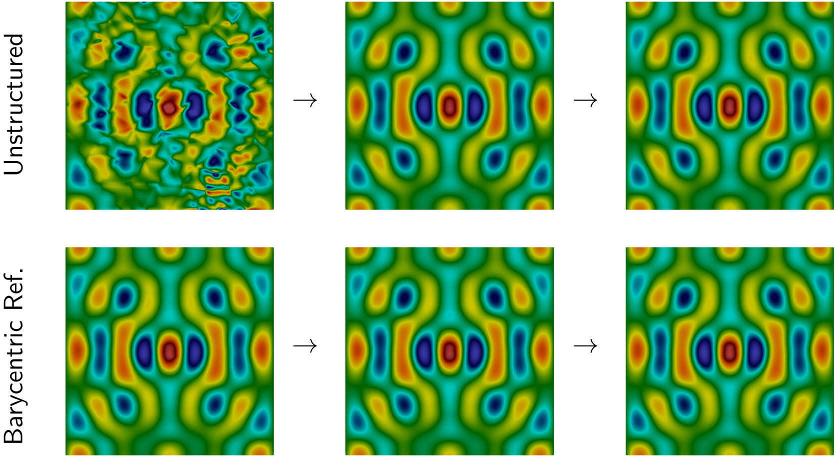

Figure: Solutions for $k = 4$.

Large Mach number

We consider the domain $\mathcal{O} = (-4, 4)^2$ and the background flow given by

$$\mathbf{b} = \frac{\alpha}{\rho} \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

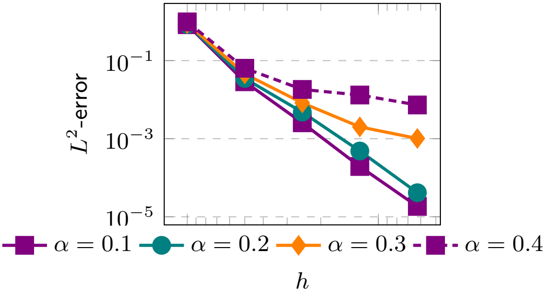

Figure: Error for different values of the coefficient in the background flow $\mathbf{b}$ and fixed polynomial order $k = 4$.

$\alpha =$	0.1	0.2	0.3	0.4
$\left\Vert c_s^{-1}\mathbf{b}\right\Vert_{\mathbf{L}^\infty} \approx$	0.22	0.31	0.38	0.44

Conclusion

Summary

- Weak T-coercivity
- Novel T-compatibility condition
- Application to Galbrun's equation

More in

M. Halla, C. Lehrenfeld, and P. Stocker, *A new T-compatibility condition and its application to the discretization of the damped time-harmonic Galbrun's equation*, arXiv preprint arxiv:2209.01878, (2022)

T. Alemán, M. Halla, C. Lehrenfeld, and P. Stocker, *Robust finite element discretizations for a simplified Galbrun's equation*, arXiv preprint arxiv:2205.15650, (2022)

M. Halla and T. Hohage, *On the well-posedness of the damped time-harmonic Galbrun equation and the equations of stellar oscillations*, SIAM J. Math. Anal., 53 (2021), pp. 4068–4095